

Homological stability 2: Arnol'd–Segal filtration (talk 4)

Configuration spaces summer school

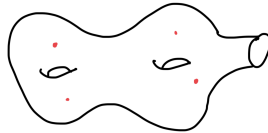
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1 Set up

Let M be a connected open manifold and $\dim(M) = q$. Let S_n denote the permutation group. Define the *Unordered configuration spaces*

$$C_n(M) = \{(x_1, \dots, x_n) \mid x_i \in M \text{ for all } i, x_i \neq x_j \text{ for all } i \neq j\} / S_n.$$

The following picture represents an element in $C_4(M)$.



Remark 1.1. $C_n(M)$ is orientable if and only if one of the following conditions hold:

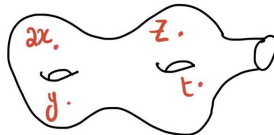
- $n = 0$,
- $\dim(M) = 1$,
- $n = 1$ and M is orientable,
- $\dim(M)$ is even and M is orientable.

For the rest of this talk, we assume that q is even and M is orientable.

Define the *Symmetric product*

$$A_n(M) = \{(x_1, \dots, x_n) \mid x_i \in M \text{ for all } i\} / S_n.$$

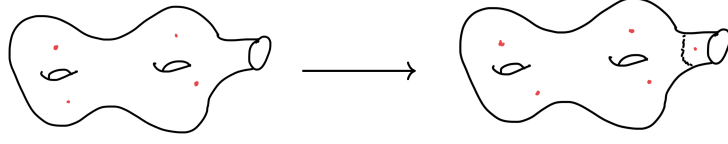
The following picture represents an element in $A_5(M)$.



Remark 1.2.

- S_n acts freely and properly discontinuously on $\{(x_1, \dots, x_n) \mid x_i \in M \text{ for all } i, x_i \neq x_j \text{ for all } i \neq j\}$, and so $C_n(M)$ is a manifold of dimension nq .
- S_n does not act freely on $\{(x_1, \dots, x_n) \mid x_i \in M \text{ for all } i\}$, and so $A_n(M)$ may not be a manifold. And $H_c^i(A_n) = 0$ for $i > nq$.

We consider the stabilization map $C_n(M) \rightarrow C_{n+1}(M)$. Informally, this map pushes the configuration away from an end of M , then adds one new point near infinity.



The goal of this talk is to prove the following theorem.

Theorem A. *The map $C_n(M) \rightarrow C_{n+1}(M)$ is a homology equivalence up to dimension $\lfloor \frac{n}{2} \rfloor$.*

Since M is an open manifold of dimension q , there exists an embedding

$$\iota: \mathbb{R}^q \sqcup M \rightarrow M$$

whose image is an open subset of M , that lies entirely in the collar neighborhood. Define the open embedding

$$\begin{aligned} F_n: \mathbb{R}^q \times C_n(M) &\rightarrow C_{n+1}(M) \\ (r, x_1, \dots, x_n) &\mapsto (\iota(r), \iota(x_1), \dots, \iota(x_n)). \end{aligned}$$

[Theorem A](#) is equivalent to F_n being a homology equivalence up to dimension $\lfloor \frac{n}{2} \rfloor$, which is equivalent by Poincaré duality to the following statement.

Proposition 1.3. *The map $F_n: \mathbb{R}^q \times C_n(M) \rightarrow C_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.*

Similarly to the definition of F_n , define the open embedding

$$G_n: \mathbb{R}^q \times A_n(M) \rightarrow A_{n+1}(M).$$

In the proof, we will assume the following statement for symmetric products $A_n(M)$, which will be proved later in talk 6.

Proposition 1.4. *The open embedding $G_n: \mathbb{R}^q \times A_n(M) \rightarrow A_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.*

2 Proof of [Theorem A](#)

Lemma 2.1. *Assume $f: Y \rightarrow Z$ is a compactly supported cohomology equivalence above the dimension m , and $H_c^i(X) \cong 0$ for $i > d$. Then the map*

$$X \times Y \rightarrow X \times Z$$

is a compactly supported cohomology equivalence above the dimension $m + d$.

Proof. By Künneth formula,

$$H_c^i(X \times Y) \cong \bigoplus_{a+b=i} H_c^a(X) \otimes H_c^b(Y), \quad \text{and } H_c^i(X \times Z) \cong \bigoplus_{a+b=i} H_c^a(X) \otimes H_c^b(Z).$$

Since $H_c^i(X) \cong 0$ for $i > d$, then we can assume that $a \leq d$. Thus $b = i - a \geq i - d$.

Let $i > m + d$. Then $b > m$, and so the map

$$f_*: H_c^b(Y) \rightarrow H_c^b(Z)$$

is an isomorphism.

Let $i = m + d$. Then $b = m$, and so f_* is surjective. We conclude that

$$X \times Y \rightarrow X \times Z$$

is a compactly supported cohomology equivalence above the dimension $m + d$. □

Notation 2.2. For every element in $A_n(M)$, we write

$$\{x_1, \dots, x_n\} = x_1 + \dots + x_n.$$

Remark 2.3. Let $\xi \in A_n(M)$, then

$$\xi = 2T + U$$

for some $T \in A_k(M), U \in C_{n-2k}(M)$ where U has multiplicity at most 1.

Example 2.4. $\{x, x, y, y, y, z\} = 2(x) + (3y + z) = 2(x + y) + (y + z)$.

Definition 2.5 (Arnol'd–Segal filtration). By Remark 2.3, every $\xi \in A_n(M)$ may be written (non-uniquely) as $\xi = 2T + U$ for some $T \in A_k(M), U \in C_{n-2k}(M)$. Define

$$A_{n,k}(M) = \{\xi = 2T + U \in A_n(M) \mid \deg(T) \geq k\}.$$

Then

$$A_{n,k}(M) \supset A_{n,k+1}(M) \quad \text{for all } k \geq 0.$$

Moreover, for every $\xi = 2T + U \in A_{n,k}(M)$,

$$n = \deg(\xi) \geq 2 \deg(T) \geq 2k.$$

Thus $A_{n,k}(M) = \emptyset$ for all $k > \lfloor \frac{n}{2} \rfloor$. We then obtain the descending sequence (known as the Arnol'd–Segal filtration)

$$A_n(M) = A_{n,0}(M) \supset A_{n,1}(M) \supset \dots \supset A_{n,\lfloor \frac{n}{2} \rfloor}(M) \supset \emptyset.$$

Example 2.6. $2x + 3y + z \in A_{6,2} \subset A_{6,1}$.

Proof of Theorem A. Recall that Theorem A is equivalent to proving that the map $F_n: \mathbb{R}^q \times C_n(M) \rightarrow C_{n+1}(M)$ is a compactly supported cohomology equivalence above dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$. We have the following identity

$$A_{n,k}(M) - A_{n,k+1}(M) = \{\xi = 2T + U \mid \deg(T) = k\} = A_k(M) \times C_{n-2k}(M). \quad (2.1)$$

Note that $A_k(M) \times C_{n-2k}(M)$ is an open subset of $A_{n,k}(M)$. We will now show by induction on n that the map F_n is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.

For $n = 1$, $F_1: \mathbb{R}^q \times C_1(M) \rightarrow C_2(M)$. We have

$$\dim(\mathbb{R}^q \times C_1(M)) = \dim(C_2(M)) = 2q,$$

and

$$(n+1)q - \lfloor \frac{n}{2} \rfloor = 2q - \lfloor \frac{1}{2} \rfloor = 2q.$$

Since $H_c^{2q}(\mathbb{R}^1 \times C_1(M)) \cong H_c^{2q}(C_2(M)) \cong \mathbb{Z}$, then

$$H_c^i(\mathbb{R}^1 \times C_1(M)) \cong H_c^i(C_2(M)) \quad \text{for all } i \geq 2q.$$

Let $n > 1$. Assume that for all $m < n$, F_m is a compactly supported cohomology equivalence above the dimension $(m+1)q - \lfloor \frac{m}{2} \rfloor$. Consider the open embedding induced by G_n ,

$$\begin{aligned} G_{n,k}: \mathbb{R}^q \times A_{n,k}(M) &\rightarrow A_{n+1,k}(M) \\ (r, \xi = 2T + U) &\mapsto \iota(r) + \iota(\xi) = 2\iota(T) + (\iota(r) + \iota(U)) \end{aligned}$$

Claim. For $k > 0$, $G_{n,k}$ is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$.

We postpone the proof of the claim till the end and we proceed with the induction proof assuming the claim. (2.1) gives that

$$A_n(M) - A_{n,1}(M) = A_{n,0}(M) - A_{n,1}(M) = A_0(M) \times C_n(M) = C_n(M).$$

Consider the long exact sequences of compactly supported cohomology,

$$\begin{array}{ccccccc}
\cdots \leftarrow H_c^i(\mathbb{R}^q \times A_{n,1}(M)) & \leftarrow & H_c^i(\mathbb{R}^q \times A_n(M)) & \leftarrow & H_c^i(\mathbb{R}^q \times C_n(M)) & \leftarrow & H_c^{i-1}(\mathbb{R}^q \times A_{n,1}(M)) \leftarrow \cdots \\
& \downarrow (G_{n,1})_* & & \downarrow (G_n)_* & & \downarrow (F_n)_* & & \downarrow (G_{n,1})_* \\
\cdots \leftarrow H_c^i(A_{n+1,1}(M)) & \leftarrow & H_c^i(A_{n+1}(M)) & \leftarrow & H_c^i(C_{n+1}(M)) & \leftarrow & H_c^{i-1}(A_{n+1,1}(M)) \leftarrow \cdots
\end{array}$$

By the claim, the maps $(G_{n,1})_*$ on the left and right are isomorphisms for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

By [Proposition 1.4](#), the map $(G_n)_*$ is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

It follows from the Five Lemma that

$$(F_n)_*: H_c^i(\mathbb{R}^q \times C_n(M)) \longrightarrow H_c^i(C_{n+1}(M))$$

is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$. □

We now turn to prove the claim.

Proof of the claim. This is a proof by downwards induction on k .

For $k > \lfloor \frac{n+1}{2} \rfloor$, $A_{n,k}(M) = A_{n+1,k}(M) = \emptyset$.

Let $k \leq \lfloor \frac{n+1}{2} \rfloor$. Assume that for all $m > k$, $G_{n,m}$ is a compactly supported cohomology equivalence above the dimension $(n+1)q - \lfloor \frac{n}{2} \rfloor$. We have by [\(2.1\)](#) that

$$\mathbb{R}^q \times A_{n,k}(M) - \mathbb{R}^q \times A_{n+1,k}(M) = A_k(M) \times C_{n-2k}(M).$$

Consider the long exact sequences of compactly supported cohomology,

$$\begin{array}{ccccccc}
\cdots \leftarrow H_c^i(\mathbb{R}^q \times A_{n,k+1}(M)) & \leftarrow & H_c^i(\mathbb{R}^q \times A_{n,k}(M)) & \leftarrow & H_c^i(A_k(M) \times (\mathbb{R}^q \times C_{n-2k}(M))) & \leftarrow & H_c^{i-1}(\mathbb{R}^q \times A_{n,k+1}(M)) \leftarrow \cdots \\
& \downarrow (G_{n,k+1})_* & & \downarrow (G_{n,k})_* & & \downarrow (F_{n-2k})_* & & \downarrow (G_{n,k+1})_* \\
\cdots \leftarrow H_c^i(A_{n+1,k+1}(M)) & \leftarrow & H_c^i(A_{n+1,k}(M)) & \leftarrow & H_c^i(C_{n-2k+1}(M)) & \leftarrow & H_c^{i-1}(A_{n+1,k+1}(M)) \leftarrow \cdots
\end{array}$$

By induction hypothesis on k , the maps $(G_{n,k+1})_*$ on the left and right are isomorphisms, for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$.

Since $k > 0$ and $n - 2k < n$, the map

$$H_c^i(\mathbb{R}^q \times C_{n-2k}(M)) \rightarrow H_c^i(C_{n-2k+1}(M))$$

is an isomorphism by the induction hypothesis on n for $i > (n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor$ and surjective for $i = (n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor$.

Since $\dim(A_k(M)) = kq$, this follows by [Lemma 2.1](#) that

$$H_c^i(A_k(M) \times (\mathbb{R}^q \times C_{n-2k}(M))) \rightarrow H_c^i(A_k(M) \times C_{n-2k+1}(M))$$

is an isomorphism by the induction hypothesis on n for $i > (n-k+1)q - \lfloor \frac{n}{2} \rfloor + k$ and surjective for $i = (n-k+1)q - \lfloor \frac{n}{2} \rfloor + k$.

It follows from the Five Lemma that

$$(G_{n,k})_*: H_c^i(\mathbb{R}^q \times A_{n,k}(M)) \longrightarrow H_c^i(A_{n+1}(M))$$

is an isomorphism for $i > (n+1)q - \lfloor \frac{n}{2} \rfloor$ and surjective for $i = (n+1)q - \lfloor \frac{n}{2} \rfloor$. □